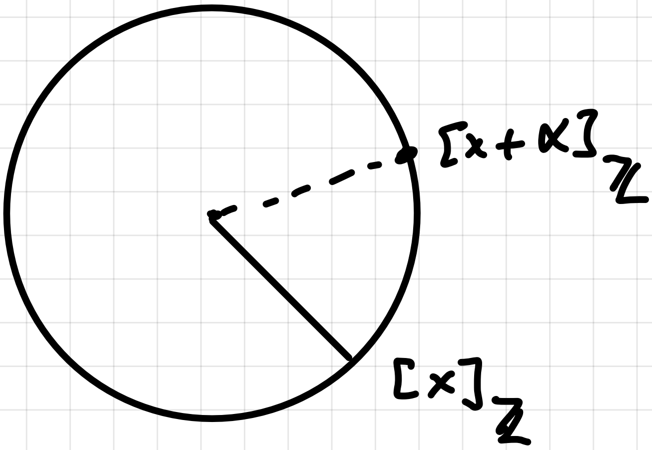
Day 7

Recall A subset $A \subset \mathbb{R}$ is *dense* if for every interval (a,b), then $(a,b) \cap A \neq \emptyset$

Back to circle rotations..

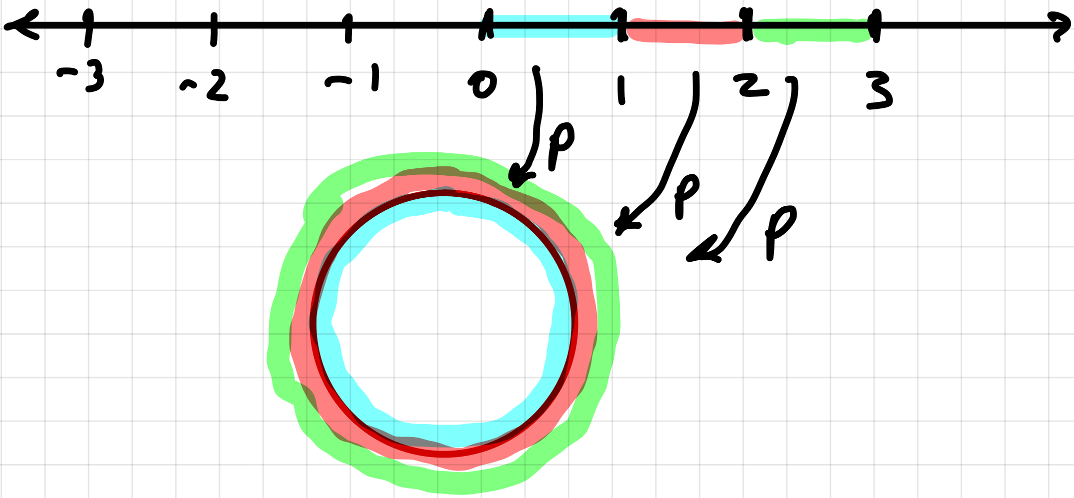


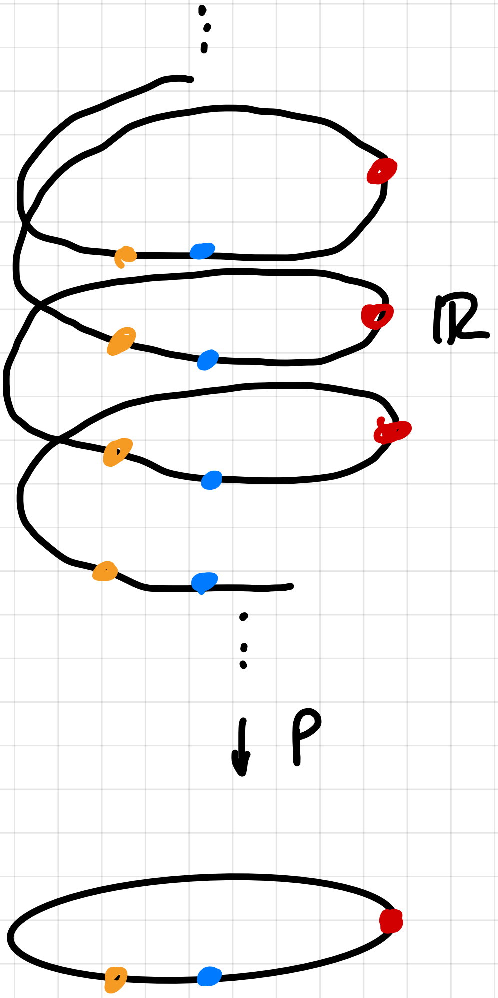
What does it mean for an orbit (seq) to be dense on the circle? Still want no gaps...

Define $p: \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, $\quad p(x) = [x]_{\mathbb{Z}}$

$$x \longmapsto [x]_{\mathbb{Z}}$$

Exercise: Show that "function" $f([x]_{\mathbb{Z}}) = x$ is <u>not</u> well defined (not posible)





p is called the projection from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$

If $A \subset \mathbb{R}/\mathbb{Z}$ let

$$p^{-1}(A) := \{x \in \mathbb{R} : p(x) \in A\}$$

"The gaps in A are the same as the gaps in $p^{-1}(A)$"
But not viceversa

Exercise Show that $\mathbb{Q}/\mathbb{Z}$ is dense in $\mathbb{R}/\mathbb{Z}$
where $\mathbb{Q}/\mathbb{Z} = \{[p/q]_{\mathbb{Z}} : \frac{p}{q} \in \mathbb{Q}\}$

Exercise Show that any finite subset of $\mathbb{R}/\mathbb{Z}$ is <u>not</u> dense in $\mathbb{R}/\mathbb{Z}$.

Recall of definition of group

A group is a set G with an operation $*$ such that $\forall g,h \in G$ then $g*h \in G$ and:

1) $\exists e \in G$ s.t. $\forall g \in G$, $e*g = g*e = g$ (identity)
2) $\forall g,h,k \in G$ s.t. $(g*h)*k = g*(h*k)$ (associativ)
3) $\forall g \in G$ $\exists g^{-1} \in G$ s.t. $g*g^{-1} = g^{-1}*g = e$ (inverse)

Example: Odd integers are not a subgroup, $0 \notin$ odds

Example: $\mathbb{Q} \subset \mathbb{R}$ is a subgroup

Example: $\{0, 1, 2, \dots\}$ is NOT a subgroup, since there are no inverses

Theorem: Let $R_\alpha : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the dynamics of rotation by α.
Then $\rho^{-1}(\mathcal{O}_\pm([0]_\mathbb{Z}))$ is a subgroup of $\mathbb{R}$,
where $\mathcal{O}_\pm([0]_\mathbb{Z}) = \{R_\alpha^n([0]_\mathbb{Z}) : n \in \mathbb{Z}\}$.

Proof: Denote $H := \rho^{-1}(\mathcal{O}_\pm([0]_\mathbb{Z}))$

Step 1.

we claim that
$$H = \{k + l\alpha : k, l \in \mathbb{Z}\}.$$

Proof of claim: Suppose $x \in H$ i.e $x \in \bar{p}^{1}(\Theta_{\pm}([0]_{\mathbb{Z}}))$

by definition of set pre-image

since $x \in p^{-1}(\Theta_{\pm}([0])_{\mathbb{Z}})$

$$p(x) = [x]_{\mathbb{Z}} \in \Theta_{\pm}([0]_{\mathbb{Z}})$$

then there exists $l \in \mathbb{Z}$ s.t

$$[x]_{\mathbb{Z}} = R_{\alpha}^{l}([0]_{\mathbb{Z}})$$

Now $R_{\alpha}^{l}([0]_{\mathbb{Z}}) = [0 + l\alpha]_{\mathbb{Z}} = [l\alpha]_{\mathbb{Z}}$

So $[x]_{\mathbb{Z}} = [l\alpha]_{\mathbb{Z}}$

this means that there exists $k \in \mathbb{Z}$ s.t

$$x = k + l\alpha.$$

This proves $x = k + l\alpha \in H = \{k + l\alpha : k, l \in \mathbb{Z}\}$

i.e $\bar{p}^{1}(\Theta_{\pm}([0])_{\mathbb{Z}}) = H \subset \{k + l\alpha : k, l \in \mathbb{Z}\}$

Exercise prove that $\{k + l\alpha : k, l \in \mathbb{Z}\} \subset \bar{p}^{1}(\Theta_{\pm}([0])_{\mathbb{Z}})$

Step 2 Now we prove that H is a subgroup.

Check: $(k + l\alpha) + (m + n\alpha) = (m+k) + (l+n)\alpha \in H$

Check: $0 \in H$? yes $0 = 0 + 0 \cdot \alpha \in H$

Check: let $x \in H$, is $-x$ in H?

yes, if $x = k + l\alpha$ for $k, l \in \mathbb{Z}$

$$-x = -(k + l\alpha) = -k + (-l)\alpha \in H$$